Weak Dirac bracket construction and the superparticle covariant quantization problem

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Abstract

The general procedure of constructing a consistent covariant Dirac-type bracket for models with mixed first and second class constraints is presented. The proposed scheme essentially relies upon explicit separation of the initial constraints into infinitely reducible first and second class ones (by making use of some appropriately constructed covariant projectors). Reducibility of the second class constraints involved manifests itself in weakening some properties of the bracket as compared to the standard Dirac one. In particular, a commutation of any quantity with the second class constraints and the Jacobi identity take place on the second class constraints surface only. The developed procedure is realized for $N = 1$ Brink–Schwarz superparticle in arbitrary dimension and for $N = 1, D = 9$ massive superparticle with Wess–Zumino term. A possibility to apply the bracket for quantizing the superparticles within the framework of the recent unified algebra approach by Batalin and Tyutin [20–22] is examined. In particular, it is shown that for $D = 9$ massive superparticle it is impossible to construct Dirac-type bracket possessing (strong) Jacobi identity in a full phase space.

1 Introduction

The superparticle covariant quantization problem has long been realized [1–4] to consist in adequate extension of the initial phase space [5]. There were a number of attempts in this direction. The most successful approaches to date are twistor-like formulations [3, 6–10], harmonic superspace technique [11–15] and the null-vectors approach of Ref. 2. Another sight to the problem lies in the fact that, instead of quantizing the original Brink–Schwarz theory, it is constructed the superparticle model [16, 17 and references therein] which will lead after quantization to the covariant SYM.

The key idea of the harmonic superspace approach [13] was to introduce additional harmonic variables, which played the role of a bridge between the $SO(1, D-1)$ indices and some internal space indices, to split the initial fermionic constraints into the first and second class parts. Then twistor-like variables might be used [3, 13] to convert the second class constraints into the first class ones, what brought the theory to the form admitting conventional canonical quantization.[1] However, the introduced auxiliary variables turn out to be various for different dimensions [3, 12]. The resulting constraint system can be irreducible or infinitely reducible depending on the dimension [3]. In the latter case there arises an additional serious problem being connected with constructing a functional integral for the model [2, 3]. Within the framework of operator quantization, the wave functions depend on twistor or harmonic variables what makes understanding the results in terms of ordinary (super)fields difficult (in the special case of a compact Lorentz-harmonic superspace, however, the problem can be solved [12]). Additional source of difficulties lies in the general status of the conversion method itself. Actually, although the approach is known for a long time [27], it still remains unclear whether a system after conversion is always physically equivalent to the original one. The general formalism may now offer a proof of the equivalence which is essentially local [28]. Such a consideration is enough for the case of a conventional perturbative field theory, but it is likely unable to take into account the effect of the reduced phase space global geometry which may have a significant influence on a physical spectrum of the particle model. In view of the all mentioned problematic points of the conversion method it seems interesting to study an alternative approach dealing with the superparticle in its original formulation [5]. In this connection, it is relevant to mention the "unified algebra" approach recently developed [20–22] just for the systems with mixed first- and second-class constraints. In principle, the proposed construction does not require an explicit separation of the first and second class constraints (what is just the basic problem of the superparticle, superstring models). An application of the procedure for concrete theories, however, implies the existence of some classical bracket (with *all* the rank and algebraic properties of the standard Dirac one) as a boundary condition to the basic generating equations [21]. Although the general construction does not include an algorithm of building this bracket, it is implied to be known "from the outset".

In the present paper we propose the general scheme of constructing a consistent covariant Dirac-type bracket for models with mixed first and second class constraints. A possibility to apply this bracket in the context of the quantization method developed in Refs. 20–22 is examined in the work.

There are two natural ways of building the bracket with needed properties. First of them consists in splitting the initial constraints into infinitely reducible first and second class parts (by making use of some covariant projectors) and subsequent generalizing the standard Dirac bracket construction to the case of infinitely reducible second class constraints. The second line is to write down the most general ansatz for the bracket and

[1] We mostly discuss $N = 1, D = 10$ case for which manifestly covariant quantization is the principal

then to require all needed rank and algebraic properties for the construction (what will specify the coefficient functions of the ansatz).

Possibilities to construct the brackets of the first kind for the superparticle, superstring models were examined in Refs. 2, 11, 18, 19, and 25. It seems surprising but the Jacobi identity is problematic for each of the suggested brackets. Actually, within the framework of the adopted scheme [2, 4] the only property[2] to be satisfied by the bracket was a weak commutation of any quantity with the second class constraints involved, i.e., only rank conditions were taken into account. Among the algebraic properties, the graded symmetry, linearity and differentiation took place by the construction while the Jacobi identity was not embedded into the scheme in some special way. In the general case, a bracket of such a kind may possess the identity in a strong sense (off the constraint surface), weakly (on the constraint shell) or has no the property at all. For instance, the Jacobi identity for the bracket of Ref. 19 takes place on the second class constraint surface only (see subsec. 3.2) and it fails to be fulfilled off-shell. As will be shown (Secs. 2 and 4) the weakening the identity is an essential ingredient of the consistent covariant Dirac-type bracket associated with the infinitely reducible second class constraints.

In this context, the question of the bracket with the strong Jacobi identity arises naturally. It turns out that it is the second approach to the building the bracket which allows to investigate the question completely. As an example, we consider $N = 1, D = 9$ massive superparticle with Wess–Zumino term [19]. In this case it proves to be possible to write down the most general ansatz for the fundamental phase variable brackets provided with the correct rank conditions (Sec. 4). Requirement of the strong Jacobi identity for the brackets implies some restrictions on the coefficient functions of the ansatz. Our result here is partly surprising. As will be show, there arises a contradictory system of equations for the coefficients if the latter were taken in the Poincaré covariant form, i.e. a Poincaré covariant closing the identity is impossible. Thus the weakening some properties of the bracket seems to be an essential ingredient of the covariant description when dealing with the models possessing mixed first and second class constraints.

The paper is organized as follows. In Sec. 2 the general procedure of building a covariant Dirac-type bracket for the models with mixed first and second class constraints is presented. Constructing such a bracket turns out to be equivalent to solving certain system of matrix equations in the enlarged phase space. As will be shown reducibility of the second class constraints (which is a price paid for the explicit covariance) manifests itself in weakening some properties of the bracket as compared to the standard Dirac one. Namely, a commutation of any quantity with the second class constraints and the Jacobi identity take place on the second class constraints surface only (note that this weakening still is compatible with the Dirac's prescriptions of quantization).

In Sec. 3 the proposed construction is realized for $N = 1$ Brink–Schwarz superparticle in arbitrary dimension, and for $N = 1, D = 9$ massive superparticle with Wess–Zumino term. In the first case the original phase space must be enlarged to include one vector

[2] Correct elimination of the redundant second class constraints implies as well linear independence of

variable (and its conjugate momentum) only. In the second case the bracket can be formulated in the initial phase space and our general construction reproduces here the results of Ref. 19. It is interesting to note that within the framework of the developed scheme the manifestly covariant (redundant) gauge fixing present no a special problem.

In Sec. 4 the constructed brackets are examined in the context of the quantization procedure of Refs. 20–22. A possibility to continue the Jacobi identity off the constraint surface (what is an essential ingredient of the classical counterpart of the quantum bracket in Ref. 21) is considered. As will be shown for the particular example of $D=9$ massive superparticle, a Poincaré covariant extension of the bracket up to one with the strong Jacobi identity is impossible. Thus, although the classical boundary conditions for the unified constraint dynamics being applied to the superparticle model can be constructed, covariant quantum realization of the quantities turns out to be problematic. In the Conclusion we summarize our results.

2 General construction

The essential ingredient of the both superparticle [5] and superstring [23] theories is Siegel local fermionic symmetry [24], which eliminates unphysical degrees of freedom and provides absence of negative norm states in the quantum spectrum of the models [2]. The constraint system of the theories in the Hamiltonian formalism includes some set of bosonic first class constraints, which we denote as $T_A \approx 0$ (A is a condensed index) as well as fermionic constraints $\chi_\alpha \approx 0$, $\alpha = 1, \ldots, n$, from which half is the first class (the generators of the Siegel transformations) and another half is second class, i.e., $\{\chi_\alpha, \chi_\beta\} \equiv \Delta^*_{\alpha\beta}$ and $\operatorname{rank} \Delta^*|_{T,\chi\approx 0} = n/2$. The symbol $\{A, B\} \equiv \overleftarrow{\partial}_i A \omega^{ij} \overrightarrow{\partial}_j B$ is used to denote the canonical Poisson bracket on a phase supermanifold, where $\overrightarrow{\partial}_i \equiv \overrightarrow{\partial}/\partial\Gamma^i$ and Γ^i are the local coordinates.

The main obstacle for operator covariant quantization of the models lies in the fact that in the original phase space it is impossible to separate the mixed first and second class constraints $\chi_\alpha \approx 0$ in a covariant irreducible way [1, 2] while the redundant splitting may present a nontrivial task [2, 18]. To avoid this difficulty we enlarge the initial phase space Γ up to $\Gamma^* \equiv (\Gamma, \Gamma_{\rm add})$, where $\Gamma_{\rm add}$ is some set of additional variables. The new variables are implied to be pure gauge and, consequently, there must be constraints eliminating $\Gamma_{\rm add}$ part of Γ^*. Denote first and second class constraints of such a kind as $\varphi_{A_1}(\Gamma^*) \approx 0$ and $\psi_{A_2}(\Gamma^*) \approx 0$, $A_1 = 1, \ldots, n_1$, $A_2 = 1, \ldots, n_2$, respectively. In what follows, we admit these constraints to be linearly independent. Further, in the extended phase space we suppose the existence of a pair of (strong) projectors $p_\alpha^{\pm\beta}(\Gamma^*)$

$$\begin{aligned} &{p^+}^2 = p^+, \qquad &{p^-}^2 = p^-, \\ &p^+ p^- = 0, \qquad &p^+ + p^- = 1 \end{aligned} \tag{2.1}$$

splitting the original mixed constraints χ_α into redundant first and second class pieces

$\chi^+ \equiv p^+\chi$ and $\chi^- \equiv p^-\chi^-$

$$\begin{array}{ll} \{\chi^+_\alpha, \chi^+_\beta\} \approx 0, & \{\chi^+_\alpha, \chi^-_\beta\} \approx 0, \\ \{\chi^-_\alpha, \chi^-_\beta\} \equiv \Delta_{\alpha\beta} \approx (p^-\Delta^* p^-)_{\alpha\beta}. & \end{array} \tag{2.2}$$

From Eqs. (2.2) it follows the defining equation for the projector operators

$$\Delta^* p^+ \approx 0. \tag{2.3}$$

Several remarks are relevant here. First, in consequence of the identities

$$p^+\chi^- \equiv 0, \qquad p^-\chi^+ \equiv 0 \tag{2.4}$$

there are only half linearly independent constraints among the conditions $\chi^\pm_\alpha \approx 0$. This means as well that the criterion of Ref. 4 for consistent elimination of redundant second class constraints is automatically satisfied within the framework of our construction. Secondly, a proof of the equivalence between the separated first and second class constraints and the initial mixed constraint system is evident (from $\chi \approx 0$ it follows $\chi^+ \approx 0$, $\chi^- \approx 0$, and vice versa). Thirdly, $\operatorname{rank} \Delta|_{T,\chi,\varphi,\psi\approx 0} = n/2$ (as a consequence of Eq. (2.3)) what correctly reflects reducibility of the resulting second class constraints. Note as well that noncovariant projectors may always be constructed by making use of initial phase space variables only. Actually, for the models concerned one can find $n/2$ linearly-independent (weak) null-vectors for the matrix Δ^*: $\Delta^*_{\alpha\beta} C^\beta{}_a(\Gamma) \approx 0$, $a = 1, \ldots, n/2$. Then the following algebraic system of equations $Q^a{}_\gamma C^\gamma{}_b = \delta^a{}_b$ can always be solved for unknown $Q^a{}_\gamma(\Gamma)$. The quantities $p_+{}^\alpha{}_\beta \equiv C^\alpha{}_a Q^a{}_\beta$, $p_-{}^\alpha{}_\beta \equiv \delta^\alpha{}_\beta - C^\alpha{}_a Q^a{}_\beta$ prove to be the needed noncovariant projectors. The task of finding the covariant projectors is less trivial, some examples will be considered in Sec. 3.

The next step of the construction is building the generalized Dirac bracket which would allow correct elimination of the redundant second class constraints (i.e., which is compatible with setting all the second class constraints strongly to zero). For this aim let us find the (symmetric) matrix $\tilde{\Delta}^{\alpha\beta}$ which is inverse to $\Delta_{\alpha\beta}$ in the following sense:

$$\Delta\tilde{\Delta} = p^-, \tag{2.5}$$

and suppose the conditions for the second class constraints $\psi_{A_2} \approx 0$:

$$\{\chi^-_\alpha, \psi_{A_2}\} = 0, \qquad \{\chi^+_\alpha, \psi_{A_2}\} \approx 0 \tag{2.6}$$

to be satisfied.[3] On the basis of these assumptions one can write down the following ansatz for the Dirac-type bracket:

$$\{A, B\}_D = \{A, B\} - \{A, \chi^-_\alpha\}\tilde{\Delta}^{\alpha\beta}\{\chi^-_\beta, B\} - \{A, \psi_{A_2}\}\tilde{\nabla}^{A_2C_2}\{\psi_{C_2}, B\}, \tag{2.7}$$

where $\tilde{\nabla}$ is the inverse matrix to $\nabla_{A_2B_2} = \{\psi_{A_2}, \psi_{B_2}\}$, $\tilde{\nabla}\nabla = 1$. For concrete models the brackets like Eq. (2.7) were also considered in Refs. 2, 11, 18, 19, and 25. They differ

by the choice of the constraint set φ_{A_1}, ψ_{A_2}, (i.e., Γ_{add}) and by the form of the operator extracting the second class constraints. Note that we require the operators splitting the first and second class constraints to be strong projectors, what ensures the equivalence of the separated and initial constraint systems. The form of Eq. (2.5) is also crucial in our approach. Let us briefly discuss the basic properties of the bracket. First, any quantity A (weakly) commutes with the second class constraints under the bracket (2.7)

$$\{A, \psi_{B_2}\}_D = 0, \qquad \{A, \chi^-_\alpha\}_D = \{A, p^-_\alpha{}^\beta\}\chi^-_\beta \approx 0. \tag{2.8}$$

On the geometrical language it means that the matrix constructed from the fundamental brackets

$$p^{ij} \equiv \{\Gamma^{*i}, \Gamma^{*j}\}_D = \omega^{ij} - \omega^{ik}\overrightarrow{\partial}_k\chi^-_\alpha\tilde{\Delta}^{\alpha\beta}\overleftarrow{\partial}_s\chi^-_\beta\omega^{sj} - \omega^{ik}\overrightarrow{\partial}_k\psi_{A_2}\tilde{\nabla}^{A_2B_2}\overleftarrow{\partial}_s\psi_{B_2}\omega^{sj} \tag{2.9}$$

is degenerate ($\mathrm{corank}\, p^{ij} = n_2 + n/2$) and the weak eigenvectors corresponding to zero eigenvalue

$$p^{ij}\partial_j\psi_{A_2} = 0, \qquad p^{ij}\partial_j\chi^-_\alpha \approx 0 \tag{2.10}$$

are normals to the second class constraints surface. It is interesting to note that the latter relation in Eq. (2.10) can be strengthened to yield a strict equality

$$p^{ij}p^-_\alpha{}^\beta\partial_j\chi^-_\beta = 0. \tag{2.11}$$

Secondly, there is a natural arbitrariness in definition of the bracket (2.7): addition of any polynomial in the second class constraints to the ansatz (2.7) does not change the (weak) properties of the bracket. Thirdly, by construction the proposed bracket possesses the graded symmetry, linearity and differentiation. The only property to be discussed specially is the Jacobi identity. To clarify this question consider the graded cycle

$$(-1)^{\epsilon_i\epsilon_k}p^{il}\overrightarrow{\partial}_l p^{jk} + (-1)^{\epsilon_i\epsilon_j}p^{jl}\overrightarrow{\partial}_l p^{ki} + (-1)^{\epsilon_k\epsilon_j}p^{kl}\overrightarrow{\partial}_l p^{ij} \tag{2.12}$$

which identically vanishes in the case of the ordinary Dirac bracket and provides the Jacobi identity for the construction. The direct proof of the identity in that case is actually based on the fact that the matrix of the second class constraints, say $M_{\alpha\beta}$, is invertible $M_{\alpha\beta}\tilde{M}^{\beta\gamma} = \delta_\alpha{}^\gamma$ and consequently $\partial_i\tilde{M} = -\tilde{M}(\partial_i M)\tilde{M}$. In the case of infinitely reducible second class constraints we deal with equation $\Delta_{\alpha\beta}\tilde{\Delta}^{\beta\gamma} = p^-{}_\alpha{}^\gamma$, from which it follows the equality

$$\overrightarrow{\partial}_l\tilde{\Delta}^{\alpha\beta} = -\tilde{\Delta}^{\alpha\gamma}\overrightarrow{\partial}_l\Delta_{\gamma\delta}\tilde{\Delta}^{\delta\beta} + p^{+\beta}_\gamma\overrightarrow{\partial}_l\tilde{\Delta}^{\gamma\alpha} + p^{+\alpha}_\gamma\overrightarrow{\partial}_l\tilde{\Delta}^{\gamma\beta} - \overrightarrow{\partial}_l(\tilde{\Delta}^{\alpha\gamma}p^{+\beta}_\gamma). \tag{2.13}$$

The new terms in the right hand side of this expression are manifestation of reducibility of the constraints involved and they actually are the source of breaking the Jacobi identity. It can be directly verified that the identity for the bracket (2.7) is not fulfilled. We can show, however, that under some additional assumptions the Jacobi identity takes place on the second class constraint surface $\chi^-_\alpha \approx 0$. The needed suppositions turn out to be of the form

whence it follows

$$\{p_\alpha^{+\beta}, \chi_\gamma\} = 0, \qquad \{p_\alpha^{+\beta}, p_\gamma^{-\delta}\} = 0, \qquad \{p_\alpha^{+\beta}, p_\gamma^{+\delta}\} = 0; \tag{2.15}$$

$$\Delta = p^- \Delta^* p^-, \qquad p^+ \Delta = 0. \tag{2.16}$$

Note as well that in consequence of Eq. (2.5) the conditions

$$p^+ \tilde{\Delta} p^- = 0, \qquad p^- \tilde{\Delta} p^+ = 0$$

are fulfilled. This means that $\tilde{\Delta}$ can be represented in the form

$$\tilde{\Delta} = p^- \tilde{\Delta}_1 p^- + p^+ \tilde{\Delta}_2 p^+$$

where $\tilde{\Delta}_1$ is some matrix consistent with Eq. (2.5) and $\tilde{\Delta}_2$ is an arbitrary matrix. The contribution of the second term into the bracket (2.7) can always be suppressed by making use of the natural arbitrariness containing in definition of the bracket (taking into account the identity $p_\delta^{+\alpha} \overrightarrow{\partial}_n \chi_\alpha^- \equiv -(\overrightarrow{\partial}_n p_\delta^{+\alpha}) \chi_\alpha^-$ one concludes that the contribution is quadratic in the second class constraints χ^-). By this reason one can search for $\tilde{\Delta}$ in the form $\tilde{\Delta} = p^- \tilde{\Delta}_1 p^-$ and suppose that

$$p^+ \tilde{\Delta} = 0. \tag{2.17}$$

Considering now Eq. (2.12) with p^{ij} being presented in the form (2.9) and using Eq. (2.13) one can get (after straightforward but extremely tedious calculations) that

$$\begin{aligned}
&(-1)^{\epsilon_j \epsilon_k} p^{kl} \overrightarrow{\partial}_l p^{ij} + \text{cycle}(ijk) = \\
&\quad -(-1)^{\epsilon_j \epsilon_k + \epsilon_l(\epsilon_i + \epsilon_\alpha)} [\omega^{kl} - \omega^{ks} \overrightarrow{\partial}_s \chi_\rho^- \tilde{\Delta}^{\rho\gamma} \overleftarrow{\partial}_p \chi_\gamma^- \omega^{pl}] \omega^{in} \overrightarrow{\partial}_n \chi_\alpha^- [p_\delta^{+\alpha} \overrightarrow{\partial}_l \tilde{\Delta}^{\beta\delta} + \\
&\quad + \overrightarrow{\partial}_l \tilde{\Delta}^{\alpha\delta} p_\delta^{+\beta}] \overleftarrow{\partial}_m \chi_\beta^- \omega^{mj} + \text{cycle}(ijk).
\end{aligned} \tag{2.18}$$

Since

$$p_\delta^{+\alpha} \overrightarrow{\partial}_n \chi_\alpha^- \equiv -(\overrightarrow{\partial}_n p_\delta^{+\alpha}) \chi_\alpha^-$$

the graded cycle (2.18) weakly vanishes and, consequently, the bracket (2.7) possesses the Jacobi identity on the second class constraints surface $\chi_\alpha^- \approx 0$.

Thus, for building the generalized Dirac bracket, which is consistent with setting all the (reducible) second class constraints strongly to zero it is sufficiently to find a solution $p_\alpha^{\pm\beta}$, $\tilde{\Delta}^{\alpha\beta}$ of Eqs. (2.1), (2.3), (2.5), and (2.14) which is compatible with Eqs. (2.6) and (2.17). Note that although the proposed bracket allows correct elimination of the second class constraints, some of its properties (see Eqs. (2.8), (2.10), (2.11), and (2.18)) are fulfilled in a weak sense only. Reducibility of the second class constraints, thus, manifests itself in weakening some properties of the standard Dirac bracket.

3 Applications

3.1 $N = 1$ Brink–Schwarz superparticle

dynamics of the model in the original phase space is determined by the following (first-order formalism) action [5]:

$$S = \int d\tau \left(p_m \Pi^m - \frac{ep^2}{2} \right), \tag{3.1}$$

$$\Pi^m \equiv \dot{x}^m - i\theta\Gamma^m\dot{\theta}.$$

For definiteness we consider ten-dimensional case here. The results of the section, however, can be directly generalized to the case of other dimensions. We use (generalized Majorana) notations, in which θ^α is a real M-W spinor $\alpha = 1, \ldots, 16$, Dirac matrices $\Gamma^m_{\alpha\beta}$ and $\tilde{\Gamma}^{m\,\alpha\beta}$ are real, symmetric, obeying the standard algebra $\Gamma^m\tilde{\Gamma}^n + \Gamma^n\tilde{\Gamma}^m = 2\eta^{mn}$.

Passing to the Hamiltonian formalism one can find the following constraints for the model:

$$p_m - \pi_m \approx 0, \qquad \pi_{pm} \approx 0, \tag{3.2a}$$

$$p_e \approx 0, \qquad \pi^2 \approx 0, \tag{3.2b}$$

$$\chi \equiv p_\theta - i\theta\Gamma^m\pi_m \approx 0 \tag{3.2c}$$

where $(p_e, \pi_{pm}, \pi_m, p_{\theta^\alpha})$ are momenta conjugate to variables $(e, p^m, x^m, \theta^\alpha)$ respectively. The constraints (3.2a) are second class and imply that the pair (p, π_p) is pure gauge. The constraints (3.2b) are first class. Among the spinor constraints (3.2c), there are eight of first class and eight of second class as a consequence of the equations (we use the Poisson bracket of the form $\{x^n, \pi_m\} = \delta^n{}_m$, $\{\theta^\alpha, p_{\theta\beta}\} = -\delta^\alpha{}_\beta$)

$$\{\chi_\alpha, \chi_\beta\} = 2i(\Gamma^m\pi_m)_{\alpha\beta}, \tag{3.3a}$$

$$\Gamma^m\pi_m\tilde{\Gamma}^n\pi_n = \pi^2 \approx 0. \tag{3.3b}$$

Our aim now is to construct the projectors (2.1), (2.3) and the generalized Dirac bracket (2.7) for the model. Technically, the task of finding the needed projectors consists in building the matrix $K_\alpha{}^\beta$ satisfying the equations $K^2 = 1$, $\Gamma^m\pi_m K \approx -\Gamma^m\pi_m$. One can show, however, that in the initial phase space it is impossible to built the covariant object of such a kind.

To construct the needed quantity let us introduce an additional vector variable A_m and consider several trivial consequences of Eq. (3.3b)

$$\begin{aligned}
&\Gamma^m\pi_m\tilde{\Gamma}^n\pi_n\Gamma^r A_r = \pi^2\Gamma^r A_r \approx 0,\\
&\Gamma^m\pi_m\tilde{\Gamma}^n A_n\Gamma^r\pi_r = 2(\pi A)\Gamma^r\pi_r - \Gamma^n A_n\pi^2 \approx 2(\pi A)\Gamma^r\pi_r,\\
&\Gamma^m\pi_m\frac{1}{2}\Big(1 + \frac{1}{2(\pi A)}\tilde{\Gamma}^{[n}\Gamma^{r]}\pi_n A_r\Big) \equiv \Gamma^m\pi_m\tilde{p}^+ \approx 0.
\end{aligned} \tag{3.4}$$

In agreement with Eq. (2.3), the arising operator $\tilde{p}^+$ is a weak eigenvector corresponding to zero eigenvalue for the matrix $\Delta^* = 2i\Gamma^m\pi_m$ and moreover it proves to be a weak projector, i.e.,

This equation can further be strengthened to yield a strict equality and the results are

$$
\begin{gathered}
p^+ = \frac{1}{2}(1+K), \qquad p^- = \frac{1}{2}(1-K), \\
K = \frac{1}{2\sqrt{(\pi A)^2 - \pi^2 A^2}} \tilde{\Gamma}^{[n}\Gamma^{m]}\pi_n A_m, \qquad K^2 = 1.
\end{gathered} \tag{3.6}
$$

Thus, to construct the needed projector operators it is sufficiently to enlarge the initial phase space by means of adding one vector variable A_m only. To be consistent, we should then include this variable into the original Lagrangian (3.1) (in a pure gauge manner). The following action:

$$
S = \int d\tau \, p_m(A^m - i\theta\Gamma^m\dot{\theta}) + B_m(A^m - \dot{x}^m) - \frac{ep^2}{2} \tag{3.7}
$$

turns out to be suitable for this goal. In constructing this action we were enforced to introduce one more additional variable B_m. Taking into account the equations of motion for the new variables

$$
\frac{\delta S}{\delta A^m} = p_m + B_m = 0, \qquad \frac{\delta S}{\delta B^m} = A_m - \dot{x}_m = 0 \tag{3.8}
$$

it is easy to show that the model (3.7) is on-shell equivalent to the original superparticle (3.1).

The supersymmetry transformations are written now as

$$
\delta\theta = \epsilon, \qquad \delta x^m = i\epsilon\Gamma^m\theta, \qquad \delta A^m = i\epsilon\Gamma^m\dot{\theta}. \tag{3.9}
$$

Local α- and k-symmetries [5, 24] take the form

$$
\begin{aligned}
\delta_\alpha x^m &= \alpha\dot{x}^m, & \delta_\alpha\theta &= \alpha\dot{\theta}^\beta, \\
\delta_\alpha p^m &= \alpha\dot{p}^m, & \delta_\alpha e &= (\alpha e)^{\cdot}, \\
\delta_\alpha B^m &= \alpha\dot{B}^m, & \delta_\alpha A^m &= (\alpha A^m)^{\cdot};
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
\delta_k\theta &= \tilde{\Gamma}^m p_m k, & \delta_k x^m &= i\theta\Gamma^m\delta\theta, \\
\delta_k e &= 4i\dot{\theta}k, & \delta_k A^m &= (i\theta\Gamma^m\delta\theta)^{\cdot}.
\end{aligned} \tag{3.11}
$$

A complete constraint system of the model in the Hamiltonian formalism is

$$
T_A: \qquad p_e \approx 0, \qquad \pi^2 \approx 0; \tag{3.12a}
$$

$$
\chi_\alpha: \qquad p_\theta - i\theta\Gamma^m\pi_m \approx 0; \tag{3.12b}
$$

$$
\varphi_{A_1}: \qquad \pi_{Am} \approx 0; \tag{3.12c}
$$

$$
\begin{aligned}
\psi_{A_2}: \qquad & p_m - \pi_m \approx 0, \quad \pi_{pn} \approx 0, \\
& p_m + B_m \approx 0, \quad \pi_{Bn} \approx 0
\end{aligned} \tag{3.12d}
$$

where we denoted the momenta conjugate to variables $(e, x^m, p^m, \theta^\alpha, A^m, B^m)$ as

is first class and another half is second class. Note that in the gauge $A^m \approx 0$ the constraint system (3.12) precisely coincides with the Brink–Schwarz one [2, 5].

Using now the projectors (3.6) to split the spinor constraints into redundant first and second class pieces $\chi^+ \equiv \chi p^+$, $\chi^- \equiv \chi p^-$

$$\begin{array}{ll} \{\chi^+_\alpha, \chi^+_\beta\} \approx 0, & \{\chi^+_\alpha, \chi^-_\beta\} \approx 0, \\ \{\chi^-_\alpha, \chi^-_\beta\} = 2i(p^-\Gamma^m \pi_m p^-)_{\alpha\beta} \equiv \Delta_{\alpha\beta}, & \\ \{\chi^\pm_\alpha, \varphi_{A_1}\} \approx 0, & \{\chi^\pm_\alpha, \psi_{A_2}\} = 0, \\ \{\chi^\pm_\alpha, T_A\} = 0 & \end{array} \tag{3.13}$$

and converting the matrix $\Delta_{\alpha\beta}$ in accordance with Eq. (2.5)

$$\tilde{\Delta}^{\alpha\beta} = \frac{p^{-\alpha}{}_\delta(\tilde{\Gamma}^n A_n)^{\delta\sigma} p^{-\beta}{}_\sigma}{2i(\sqrt{(\pi A)^2 - \pi^2 A^2} + (\pi A))}, \qquad \tilde{\Delta}\Delta = p^- \tag{3.14}$$

one can write down the final expression for the generalized Dirac bracket

$$\begin{aligned} \{A, C\}_D = \{A, C\} - \{A, \chi^-_\alpha\} & \frac{(p^-\tilde{\Gamma}^n A_n p^-)^{\alpha\beta}}{2i(\sqrt{(\pi A)^2 - \pi^2 A^2} + (\pi A))} \{\chi^-_\beta, C\} - \\ -\{A, \pi_p{}^m\}\{p_m - \pi_m, C\} & + \{A, p^m - \pi^m\}\{\pi_{pm}, C\} - \{A, \pi_B{}^m\}\{p_m + B_m, C\} + \\ +\{A, p^m + B^m\}\{\pi_{Bm}, C\} & - \{A, p^m - \pi^m\}\{\pi_{Bm}, C\} + \{A, \pi_B{}^m\}\{p_m - \pi_m, C\}. \end{aligned} \tag{3.15}$$

Since the projectors (3.6) satisfy the Eqs. (2.14), the proposed bracket possesses (in a weak sense) all the rank and algebraic properties of the standard Dirac bracket. Consistent covariant elimination of the (reducible) second class constraints is now possible. The explicit form of the fundamental phase variable brackets is (we omit here the brackets corresponding to unphysical variables (p, π_p), (B, π_B))

$$\begin{aligned} &\{\theta^\alpha, \theta^\beta\}_D = \frac{i}{q}(p^- \tilde{\not{A}} p^-)^{\alpha\beta}, \\ &\{\theta^\alpha, p_{\theta_\beta}\}_D = -\delta^\alpha{}_\beta + \frac{1}{2} p^{-\alpha}{}_\beta, \\ &\{p_{\theta_\alpha}, p_{\theta_\beta}\}_D = -\frac{i}{2}(p^- \not{\pi} p^-)_{\alpha\beta}, \\ &\{x^m, \theta^\alpha\}_D = -\frac{1}{q}(\theta\Gamma^m \tilde{\not{A}} p^-)^\alpha - \frac{i}{q}\Big(\chi^+\Big[\frac{\partial}{\partial\pi_m}p^-\Big] \tilde{\not{A}} p^-\Big)^\alpha, \\ &\{x^m, p_{\theta_\alpha}\}_D = \frac{i}{2}(\theta\Gamma^m p^-)_\alpha - \frac{1}{2}\Big(\chi^+ \frac{\partial}{\partial\pi_m}p^-\Big)_\alpha, \\ &\{\pi_A{}^m, \theta^\alpha\}_D = \frac{i}{q}\Big(\chi^+\Big[\frac{\partial}{\partial A_m}p^-\Big] \tilde{\not{A}} p^-\Big)^\alpha, \\ &\{\pi_A{}^m, p_{\theta_\alpha}\}_D = \frac{1}{2}\Big(\chi^+ \frac{\partial}{\partial A_m}p^-\Big)_\alpha, \end{aligned} \tag{3.16}$$

$$\{x^m, x^n\}_D = \frac{i}{q}\theta\Gamma^m p^- \tilde{\not{A}} p^- \Gamma^n \theta - \frac{1}{q}\theta\Gamma^m \tilde{\not{A}}\Big[\frac{\partial p^-}{\partial\pi_n}\Big]\chi^+ +$$

$$
\begin{aligned}
&\{x^m, \pi_n\}_D = \delta^m{}_n, \\
&\{x^m, \pi_{A_n}\}_D = \frac{1}{q}\theta\Gamma^m \tilde{\not{A}}\Big[\frac{\partial p^-}{\partial A^n}\Big]\chi^+ + \frac{i}{q}\chi^+\Big[\frac{\partial p^-}{\partial \pi_m}\Big] \tilde{\not{A}}\Big[\frac{\partial p^-}{\partial A^n}\Big]\chi^+, \\
&\{A^n, \pi_{A_m}\}_D = \delta^n{}_m, \\
&\{\pi_{A_m}, \pi_{A_n}\}_D = -\frac{i}{q}\chi^+\Big[\frac{\partial p^-}{\partial A^m}\Big] \tilde{\not{A}}\Big[\frac{\partial p^-}{\partial A^n}\Big]\chi^+,
\end{aligned}
$$

were we denoted $q = 2(\sqrt{(\pi A)^2 - \pi^2 A^2} + (\pi A))$, $\tilde{\not{A}} = \tilde{\Gamma}^n A_n$, $\not{\pi} = \Gamma^n \pi_n$ and used the identities $p^- \not{\pi} = \not{\pi} p^-$, $p^- \tilde{\not{A}} = \tilde{\not{A}} p^-$, $p^+ \not{\pi} p^- = p^+ \tilde{\not{A}} p^- = 0$, $p^- \partial p^- p^- = 0$.

Note as well that the presented scheme allows the covariant (redundant) gauge

$$\theta^+ \equiv p^+\theta \approx 0 \tag{3.17}$$

for the fermionic first class constraints.

Let us briefly discuss a relation between the considered formulation and the Hamiltonian null-vectors approach of Ref. 2.

The basic idea of the construction proposed in Ref. 2 was to introduce two null vectors

$$n^2 = 0, \qquad r^2 = 0, \qquad nr = -1 \tag{3.18}$$

(which were considered as pure gauge variables) to separate the initial fermionic second class constraints in covariant and redundant way

$$\psi \equiv \chi \not{n} \not{r} \approx 0. \tag{3.19}$$

Note that in the presence of the constraints (3.18) the operators

$$\tilde{p}^- = \frac{1}{2(nr)} \not{n} \not{r}, \qquad \tilde{p}^+ = \frac{1}{2(nr)} \not{r} \not{n} \tag{3.20}$$

form weak projectors (after constructing the Dirac bracket associated with the full system of second class constraints [2], the constraints (3.18) can be considered as strong equations and the operators (3.20) become strong projectors). One can believe, therefore, that the new variables were introduced to construct a projector operator extracting the fermionic second class constraints.

Return now to the formulation (3.7) and let us use the variables A^m, π^n to define a pair of strong null vectors (see also Ref. 25)

$$
\begin{aligned}
n'^m &= \frac{1}{c}[A^2\pi^m - ((A\pi) + \sqrt{(A\pi)^2 - A^2\pi^2})A^m], \\
r'^m &= \frac{1}{c}[A^2\pi^m - ((A\pi) - \sqrt{(A\pi)^2 - A^2\pi^2})A^m],
\end{aligned} \tag{3.21}
$$

$$n'^2 \equiv 0, \qquad r'^2 \equiv 0, \qquad n'r' \equiv -1$$

where we denoted $c = \sqrt{2A^2((\pi A)^2 - \pi^2 A^2)}$. The crucial observation is that the following identities:

$$p^- = \frac{1}{2}\Big(1 - \frac{1}{2b}\tilde{\Gamma}^{[n}\Gamma^{m]}\pi_n A_m\Big) \equiv \frac{1}{2(n'r')} \not{n}' \not{r}', \tag{3.22}$$

where $b \equiv \sqrt{(\pi A)^2 - \pi^2 A^2}$, are fulfilled.

Thus, the basic constructions of Ref. 2 can be reproduced within the context of the theory (3.7) and, in this sense, there is a correspondence between two formulations. It should be noted, however, that the model (3.7) is free from some difficulties of Ref. 2. In particular, the constraint system for additional variables (φ_{A_1}, ψ_{A_2} in terminology of Sec. 2) in Ref. 2 is reducible and very complicated as compared to Eqs. (3.12c) and (3.12d). The operator separating the (redundant) first class constraints ($\Gamma^m \pi_m$) is not a projector and a covariant proof of the equivalence between the splitted and original mixed constraints presents a special problem. In our case this proof is evident. Note as well that the formulation of Ref. 2 is essentially Hamiltonian.

3.2 $N = 1, D = 9$ massive superparticle with Wess–Zumino term

In this subsection, as an example of the model for which the generalized bracket can be constructed in the initial phase space we consider $N = 1, D = 9$ massive superparticles with Wess–Zumino term [19].

The basic observation lies in the fact that in certain dimensions there exists Lorentz invariant, real, symmetric tensor $X_{\alpha\beta}$ (in addition to the Dirac matrices) which can be used to build the needed projectors. To prove the existence of such a tensor for the case concerned, let us construct the minimal spinor representation of $SO(1, 8)$ (which has a complex dimension $2^{(D-1)/2} = 16$) and the corresponding Γ-matrices in the explicit form.

For this aim it is sufficiently to find nine 16×16 matrices ${\Gamma^m}_\alpha{}^\beta$ satisfying the equation $\Gamma^m\Gamma^n + \Gamma^n\Gamma^m = -2\eta^{mn}$, $m = 0, 1, \ldots, 8$. Taking into account that $SO(1, 9)$-matrices from the previous section (which we denote now as ${\gamma^m}_{\alpha\beta}$, $\tilde{\gamma}^{m\alpha\beta}$) have the needed dimension, one can consider the following decomposition:

$$\begin{array}{lll} X_{\alpha\beta} \equiv {\gamma^9}_{\alpha\beta}, & \tilde{X}^{\alpha\beta} \equiv \tilde{\gamma}^{9\alpha\beta}, & {\Gamma^m}_{\alpha\beta} \equiv {\gamma^m}_{\alpha\beta}, \\ \tilde{\Gamma}^{m\alpha\beta} \equiv \tilde{\gamma}^{m\alpha\beta}, & {\Gamma^m}_\alpha{}^\beta \equiv {\Gamma^m}_{\alpha\delta}\tilde{X}^{\delta\beta}, & m = 0, 1, \ldots, 8. \end{array} \tag{3.23}$$

The properties of γ^m, $\tilde{\gamma}^m$ induce the following relations for X and Γ:

$$\begin{array}{l} X_{\alpha\beta} = X_{\beta\alpha}, \qquad X^*_{\alpha\beta} = X_{\alpha\beta}, \qquad X_{\alpha\beta}\tilde{X}^{\beta\gamma} = \delta_\alpha{}^\gamma, \\ X_{\alpha\beta}\tilde{\Gamma}^{m\beta\gamma} + {\Gamma^m}_{\alpha\beta}\tilde{X}^{\beta\gamma} = 0, \\ {\Gamma^m}_\alpha{}^\beta {\Gamma^n}_\beta{}^\gamma + {\Gamma^n}_\alpha{}^\beta {\Gamma^m}_\beta{}^\gamma = -2\eta^{mn}. \end{array} \tag{3.24}$$

Thus, the minimal spinor representation of $SO(1, 8)$ is a complex spinor ψ^α transforming, by definition, as follows

$$\delta\psi^\alpha = \frac{1}{2}\omega_{mn}(\tilde{\Gamma}^{mn})^\beta{}_\alpha\psi^\alpha, \tag{3.25}$$

where

$$\tilde{\Gamma}^{mn} = \frac{1}{4}(\tilde{\Gamma}^m\Gamma^n - \tilde{\Gamma}^n\Gamma^m).$$

Since the combination $\bar{\psi}_\alpha \equiv X_{\alpha\beta}\psi^\beta$ is transformed as $\delta\bar{\psi}_\alpha = -\frac{1}{2}\omega_{mn}(\bar{\psi}\tilde{\Gamma}^{mn})_\alpha$, we conclude

The action functional of the theory is given by the expression

$$S=\int d\tau \Big(e^{-1}\frac{\Pi^2}{2}-\frac{m^2 e}{2}+im\dot{\theta}X\theta\Big), \tag{3.26}$$

$$\Pi^m=\dot{x}^m-i\theta\Gamma^m\dot{\theta},$$

where we have introduced the einbein tangent to the superparticle wordline as opposed to the action of Ref. 19. In this formulation a form of the local symmetries becomes evident

$$\delta_\alpha x^m=\alpha\dot{x}^m, \qquad \delta_\alpha\theta^\beta=\alpha\dot{\theta}^\beta, \qquad \delta_\alpha e=(\alpha e)^{\cdot}; \tag{3.27a}$$

$$\delta_k\theta=(\tilde{\Gamma}^m\Pi_m+me\tilde{X})k, \qquad \delta_k x^m=i\theta\Gamma^m\delta\theta, \qquad \delta_k e=4ie\dot{\theta}k, \tag{3.27b}$$

The constraint system of the theory in the Hamiltonian formalism is

$$p_e\approx 0, \qquad \pi^2+m^2\approx 0, \qquad \chi\equiv p_\theta-i\theta(\Gamma^n\pi_n+mX)\approx 0, \tag{3.28}$$

where the variables $(p_e,\pi_m,p_{\theta_\alpha})$ are momenta conjugate to (e,x^m,θ^α) respectively. Dynamics of the model is governed by the Hamiltonian

$$H=p_e\lambda_e+\lambda_\theta\chi+\frac{e}{2}(\pi^2+m^2) \tag{3.29}$$

where λ_e,λ_θ are Lagrange multipliers to the constraints p_e and χ respectively. The bosonic constraints in Eq. (3.28) are first class, while there are half of first class constraints and half of second class ones among χ_α

$$\{\chi_\alpha,\chi_\beta\}=2i(\Gamma^n\pi_n+mX)_{\alpha\beta}, \tag{3.30a}$$

$$(\Gamma^m\pi_m+mX)(\tilde{\Gamma}^n\pi_n+m\tilde{X})=m^2+\pi^2\approx 0. \tag{3.30b}$$

Let us construct the generalized Dirac bracket for the model. The first step is building the projector operators satisfying Eqs. (2.1), (2.3), and (2.14). Taking into account Eqs. (3.30b) and (3.24) one can find the needed quantities (see also Ref. 19)

$$p^+=\frac{1}{2}(1+K), \quad p^-=\frac{1}{2}(1-K), \quad K_\alpha{}^\beta=\frac{1}{\sqrt{-\pi^2}}\pi_m(X\tilde{\Gamma}^m)_\alpha{}^\beta, \quad K^2=1. \tag{3.31}$$

It is straightforward to check as well that the following identities:

$$p^+\Big(X+\frac{1}{\sqrt{-\pi^2}}\Gamma^m\pi_m\Big)=0, \qquad p^-\Big(X-\frac{1}{\sqrt{-\pi^2}}\Gamma^m\pi_m\Big)=0; \tag{3.32a}$$

$$p^\pm\Gamma^m\pi_m=\Gamma^m\pi_m p^\pm, \qquad p^\pm X=Xp^\pm, \tag{3.32b}$$

$$p^+\Gamma^m\pi_m p^-=0, \qquad p^+Xp^-=0 \tag{3.32c}$$

are fulfilled.

In the presence of the projectors the fermionic constraints are splitted into (redundant) first and second class pieces $\chi^+\equiv p^+\chi$ and $\chi^-\equiv p^-\chi$:

$$\{\chi^+_\alpha, \chi^-_\beta\} = 0, \tag{3.33}$$
$$\{\chi^-_\alpha, \chi^-_\beta\} = \frac{2i}{\sqrt{-\pi^2}}(\sqrt{-\pi^2}+m)(p^-\Gamma^n\pi_n)_{\alpha\beta}$$

(π^m is supposed to be a space-like vector, therefore the constraint $\pi^2+m^2\approx 0$ is equivalent to $m-\sqrt{-\pi^2}\approx 0$). Using Eqs. (3.32) one can choose, further, more simple basis of the constraints:

$$p^+_\theta \approx 0, \quad \chi^- \equiv p^-_\theta - i\theta^- X(m+\sqrt{-\pi^2}) \approx 0, \quad m-\sqrt{-\pi^2}\approx 0, \quad p_e \approx 0 \tag{3.34}$$

where we denoted $p^\pm_\theta = p^\pm p_\theta$, $\theta^\pm = p^\pm\theta$. In this representation, finding $\tilde\Delta^{\alpha\beta}$ for Eq. (2.5) and building the generalized bracket present no a special problem. The results are

$$\{A,B\}_D = \{A,B\} - \{A,\chi^-_\alpha\}\tilde\Delta^{\alpha\beta}\{\chi^-_\beta, B\}, \tag{3.35a}$$

$$\tilde\Delta^{\alpha\beta} = \frac{(\tilde X p^-)^{\alpha\beta}}{2i(m+\sqrt{-\pi^2})}. \tag{3.35b}$$

Since the projectors (3.31) satisfy Eq. (2.14) the constructed bracket possesses (in a weak sense) all the rank and algebraic properties of the standard Dirac bracket. Thus, Eqs. (3.31), (3.35a), and (3.35b) specify the generalized Dirac bracket for the massive superparticle with Wess–Zumino term.

The explicit form of the fundamental phase variable brackets is

$$\begin{aligned}
&\{\theta^\alpha,\theta^\beta\}_D = \frac{i}{2(m+\sqrt{-\pi^2})}(\tilde X p^-)^{\alpha\beta},\\
&\{\theta^\alpha, p_{\theta_\beta}\}_D = -\delta_\beta{}^\alpha + \frac{1}{2}p^-_\beta{}^\alpha,\\
&\{p_{\theta_\alpha}, p_{\theta_\beta}\}_D = -\frac{i}{2}(m+\sqrt{-\pi^2})(Xp^-)_{\alpha\beta},\\
&\{x^m,\theta^\alpha\}_D = \frac{i}{4(m+\sqrt{-\pi^2})\sqrt{-\pi^2}}\Big(\tilde\Gamma^m + \frac{\pi^m\tilde X}{\sqrt{-\pi^2}}\Big)^{\alpha\lambda}(p^+_\theta - i\theta^+X(m+\sqrt{-\pi^2}))_\lambda +\\
&\qquad + \frac{1}{2(m+\sqrt{-\pi^2})\sqrt{-\pi^2}}\pi^m\theta^{-\alpha},\\
&\{x^m, p_{\theta_\alpha}\}_D = \frac{1}{4\sqrt{-\pi^2}}\Big(X\tilde\Gamma^m + \frac{\pi^m}{\sqrt{-\pi^2}}\Big)(p^+_\theta - i\theta^+X(m+\sqrt{-\pi^2}))_\alpha - \frac{i}{2\sqrt{-\pi^2}}\pi^m(\theta^-X)_\alpha,\\
&\{x^m,\pi_n\}_D = \delta^m{}_n,\\
&\{x^m,x^n\}_D = \frac{i}{16(m+\sqrt{-\pi^2})\pi^2}(p^+_\theta - i\theta^+X(m+\sqrt{-\pi^2}))\tilde\Gamma^{[m}X\tilde\Gamma^{n]}(p^+_\theta - i\theta^+X(m+\sqrt{-\pi^2}))-\\
&\qquad - \frac{1}{4(m+\sqrt{-\pi^2})\pi^2}\theta^-X\tilde\Gamma^{[m}\pi^{n]}(p^+_\theta - i\theta^+X(m+\sqrt{-\pi^2})),
\end{aligned} \tag{3.36}$$

were we denoted $A^{[n}B^{m]} \equiv A^nB^m - A^mB^n$ and used the identity $\left(\tilde X - \frac{1}{\sqrt{-\pi^2}}\tilde\Gamma^n\pi_n\right)p^+ \equiv 0$.

The following remarks seem to be relevant: First, taking into account Eq. (2.18) one can check that the strong Jacobi identity problem appears only in the cycles including the variable x^m. Secondly, the considered scheme admits the covariant (reducible) gauge

and the corresponding Dirac-type bracket

$$\{A,B\}_D=\{A,B\}+\{A,p_\theta^+\}p^+\{\theta^+,B\}+\{A,\theta^+\}p^+\{p_\theta^+,B\}-\{A,\chi^-\}\tilde{\Delta}\{\chi^-,B\}. \quad (3.38)$$

Thus, the physical sector of the model is exhausted by the variables $(x^m,\pi_n,\theta^{-\alpha})$ with commutation relations being presented in the form

$$\begin{aligned}
\{\theta^{-\alpha},\theta^{-\beta}\}&=-\frac{1}{2i(m+\sqrt{-\pi^2})}(p^-\tilde{X})^{\alpha\beta},\\
\{\theta^{-\alpha},x^m\}&=\frac{1}{2\sqrt{-\pi^2}}(\theta^-X\tilde{\Gamma}^m)^\alpha+\frac{1}{2}\Big(\frac{1}{\pi^2}-\frac{1}{(m+\sqrt{-\pi^2})\sqrt{-\pi^2}}\Big)\pi^m\theta^{-\alpha},\\
\{x^m,\pi_n\}&=\delta^m{}_n,\\
\{x^m,x^n\}&=-\frac{i(m+\sqrt{-\pi^2})}{4\pi^2}\theta^-\Gamma^{[m}\tilde{\Gamma}^{n]}X\theta^-.
\end{aligned} \quad (3.39)$$

It is straightforward to check now that the Jacobi identity for the brackets (3.39) is fulfilled in a strong sense. Thirdly, it was shown in Refs. 19 and 26 that appearing the Wess–Zumino term in the superparticle action plays a role of introducing a central charge into the super-Poincaré algebra. As was seen above, it was this quantity which allowed to construct the generalized Dirac bracket for the model.

4 Off-shell continuation of the generalized brackets and the unified constraint dynamics

In constructing the generalized Dirac bracket associated with the infinitely reducible second class constraints, the essential property which was embedded into the scheme was the weak Jacobi identity (the special restrictions were to be imposed to provide the property). To apply standard covariant quantization methods in a *full* phase space, it is necessary then to continue the bracket up to one with the strong Jacobi identity. Note in this context that the bracket structure is a sum of its body on the constraint surface, its soul (fermionic terms) and the constraints involved. The rank of the bracket is defined by the first terms only. One can believe that the strong Jacobi identity is absent because not all needed fermions and constraints were added to the body. We may add arbitrary (the most general) combination of such terms with some coefficient functions. Then the requirement of the strong Jacobi identity for the new bracket will fix these functions.

Another serious motivation for studying the question concerns a possibility to apply the quantization scheme by Batalin and Tyutin [20–22] to the superparticle models. The remarkable feature of the formalism developed in Refs. 20–22 lies in the fact that it, in principle, allows to avoid the explicit separation of the constraints into the first and second class ones. Let us enumerate some relevant facts. A solution of quantum generating equations (in the lowest orders) implies the following defining relations for the classical counterparts [20]

$$\{\Gamma^A, \Theta^\alpha\} = E^{A\alpha} + Y^{A\alpha}{}_\beta \Theta^\beta, \tag{4.2}$$

$$\{\Theta^\alpha, \Theta^\beta\} = U^{\alpha\beta}{}_\gamma \Theta^\gamma, \tag{4.3}$$

$$(\{\Gamma^A, D^{BC}\} + Z^{AB}{}_\alpha E^{C\alpha}(-1)^{\epsilon_\alpha \epsilon_C})(-1)^{\epsilon_A \epsilon_C} + \text{cycle}(ABC) = X^{ABC}{}_\alpha \Theta^\alpha, \tag{4.4}$$

where Γ^A are variables and Θ^α are linearly-independent constraints of a theory. The numers M' (M'') of first (second) class constraints among Θ^α, $\alpha = 1, \ldots, M' + M''$, are fixed by conditions

$$\text{rank}\, \|E^{A\alpha}\||_{\Theta=0} = M', \quad \text{corank}\, \|D^{AB}\||_{\Theta=0} = M''. \tag{4.5}$$

The quantities D^{AB}, $E^{A\alpha}$ must be embedded into the scheme from the outset as a boundary conditions for generating equations and, as it seen from Eqs. (4.3), (4.4), satisfy to weakened version of the standard properties of the Dirac bracket.

In Refs. 20–22 an existence of a solution of quantum generating equations with such boundary conditions was shown. In particular, there exists, in principle, the quantum analogue of $\hat{Z}^{AB}{}_\alpha$ supplying the Jacobi identity for fundamental brackets (4.1). In previous sections we found the quantities D and E for the concrete models. To complete the scheme at the classical level, we investigate the question of existence the *covariant* quantity $Z_\alpha{}^{AB}$. Namely, for massive superparticle considered above, it is possible, instead of continuation of the available D and E, to write down the most general Poincaré covariant ansatz for fundamental brackets (4.1) with an accuracy of some scalar coefficients. The coefficients of the body ansatz will be found from the requirement that the conditions (4.1)–(4.3), (4.5) are fulfilled. The remaining coefficients will then be fixed by demanding the strong Jacobi identity for the bracket. Thus, our considerations are not related to an existence of projectors or a particular form of the quantities D, E.

The basic observation lies in the fact that there arises a contradictory system of equations for the coefficients, and our result looks as follows: for the $D = 9$ massive superparticle it is impossible to construct a Poincaré covariant bracket obeying the conditions (4.1)–(4.5).

To prove the fact let us demonstrate first the following assertion (the fermionic variables and constraints from the subsec. 3.2 are denoted now as $(\tilde{\theta}^\alpha, p_{\theta\alpha}) \equiv Z^A$ and $\tilde{L}_\alpha$, respectively):

From the conditions

a) the constraints $\tilde{L}_\alpha \approx 0$, $T \equiv \pi^2 + m^2 \approx 0$ are in involution;

b) the rank conditions

$$\text{corank}\, \{\tilde{L}_\alpha, \tilde{L}_\beta\}|_{L=T=0} = 8, \qquad \text{rank}\{Z^A, \tilde{L}_\alpha\}|_{L=T=0} = 8 \tag{4.6}$$

are fulfilled, it follows that a body of the bracket on the constraints surface is determined in the odd-sector as

$$\{L_\alpha, L_\beta\} = O^1(T) + O^1(L) O^1(\theta, L), \tag{4.7a}$$

$$\{L_\alpha, \theta^\beta\} = -p^+{}_\alpha{}^\beta + O^1(T) p^-{}_\alpha{}^\beta + O^1(T) p^+{}_\alpha{}^\beta + O^2(\theta, L), \tag{4.7b}$$

where the brackets were written in terms of the shifted variables $(\theta^\alpha, L_\alpha)$

$$
\begin{aligned}
&\theta^\alpha = \frac{1}{2\sqrt{-imc_2}}\Big[\tilde\theta^\alpha - \frac{c_1}{2k_2}\tilde X^{\alpha\beta}(p_{\theta\beta} - i\tilde\theta^\gamma(\Gamma^m\pi_m + mX)_{\gamma\beta})\Big],\\
&L_\alpha = -\frac{2\sqrt{-imc_2}}{k_2}[p_{\theta\alpha} - i\tilde\theta^\gamma(\Gamma^m\pi_m + mX)_{\gamma\alpha}],\\
&c_2(\pi^m) \neq 0, \qquad k_2(\pi^m) \neq 0;
\end{aligned}
\tag{4.8}
$$

and we used the weak projectors

$$
p^{\pm}{}_\alpha{}^\beta \equiv \frac{1}{2}\Big(1 \pm \frac{1}{m}X\tilde\Gamma^m\pi_m\Big)_\alpha{}^\beta. \tag{4.9}
$$

The terms proportional to θ and L are denoted as $O^1(\theta, L)$; analogously we denoted $O^2(\theta, L) \equiv \theta^2 \ldots + \theta L \ldots + L^2 \ldots$. The symbol $O^1(T)$ denotes the terms linear in the bosonic constraint. The coefficients $k_i, c_i, a_i, \ldots$, arising in all expressions are scalar functions depending on the variable π^m only. Evidently, (non)existence of the bracket with the needed properties in terms of shifted variables implies the same for the initial bracket.

To prove the assertion, note that under the condition a) the most general Poincaré covariant ansatz for the brackets in the considered sector can be put into the following form (where only the body on the constraint surface is written in an explicit form)

$$
\{\tilde L_\alpha, \tilde L_\beta\} = O^1(T) + O^1(\tilde L)O^1(\tilde\theta, \tilde L), \tag{4.10a}
$$

$$
\{\tilde L_\alpha, \tilde\theta^\beta\} = k_1 p^-{}_\alpha{}^\beta + k_2 p^+{}_\alpha{}^\beta + O^2(\tilde\theta, \tilde L) + O^1(T)p^-{}_\alpha{}^\beta + O^1(T)p^+{}_\alpha{}^\beta, \tag{4.10b}
$$

$$
\{\tilde\theta^\alpha, \tilde\theta^\beta\} = c_1\tilde X p^+ + c_2(\tilde X p^-) + O^2(\tilde\theta, \tilde L) + O^1(T). \tag{4.10c}
$$

To evaluate consequences of the rank conditions (4.6) let us pass to the rest frame

$$
\begin{gathered}
\pi^m = (m, 0, \ldots, 0), \qquad \pi^2 = -m^2,\\
\Gamma^m\pi_m + mX = 2m\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde\Gamma^m\pi_m + m\tilde X = -2m\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\\
p^- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad p^+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\end{gathered}
\tag{4.11}
$$

As a consequence of these relations, the body of the bracket in the odd-sector (when restricted to the constraint surface) is

$$
\{z^A, z^B\} =
\begin{array}{cc}
 & \begin{array}{cc} \tilde L & \tilde\theta \end{array} \\
\begin{array}{c} \tilde L \\ \\ \tilde\theta \end{array} &
\left|\begin{array}{ccc|ccc}
 & & & k_1\mathbf{1}_8 & \vdots & 0 \\
 & 0 & & \cdots\cdots & \vdots & \cdots\cdots \\
 & & & 0 & \vdots & k_2\mathbf{1}_8 \\
--- & - & --- & --- & - & --- \\
k_1\mathbf{1}_8 & \vdots & 0 & & & \\
\cdots\cdots & \vdots & \cdots\cdots & & * &
\end{array}\right|
\end{array}. \tag{4.12}
$$

If $k_1 \neq 0$ and $k_2 \neq 0$ we have a nondegenerate matrix. So, to satisfy Eq. (4.6) it is necessary to assume that $k_1 = 0$ (another possibility $k_2 = 0$ can be considered along the same lines). Thus, instead of (4.10b) we have

$$\{\tilde{L}_\alpha, \tilde{\theta}^\beta\} = k_2 {p^+}_\alpha{}^\beta + O^2(\tilde{\theta}, \tilde{L}) + O^1(T){p^-}_\alpha{}^\beta + O^1(T){p^+}_\alpha{}^\beta. \tag{4.13}$$

Shifting then $\tilde{\theta}^\alpha$

$$\tilde{\theta}^\alpha \to \tilde{\theta}^\alpha - \frac{c_1}{2k_2}\tilde{X}^{\alpha\beta}\tilde{L}_\beta, \tag{4.14}$$

one can get

$$\{\theta^\alpha, \theta^\beta\} = c_2(\tilde{X}p^-)^{\alpha\beta} + O^2(\theta, L) + O^1(T). \tag{4.15}$$

Subsequent renormalizations of θ and L can further be used to eliminate the coefficients $k_2 \neq 0$ from the body of the bracket, what will reproduce Eqs. (4.7a)–(4.7c), (4.8). The structure of brackets does not change under all needed renormalizations since all the arising additional contributions are of orders $O^2(\theta, L)$, $O^1(T)$. Note that the body of the bracket (4.7) and the corresponding expressions from (3.36) are the same.

Let us add the remaining variables and write the most general Poincaré covariant ansatz for all brackets

$$\begin{aligned}
&\{\theta^\alpha, \theta^\beta\} = \underline{-\frac{(\tilde{X}p^-)^{\alpha\beta}}{4im}} + O^1(T)\tilde{X}p^+ O^1(T)\tilde{X}p^- + O^2(\theta, L) + \dots \\
&\{L_\beta, \theta^\alpha\} = \underline{-{p^+}_\beta{}^\alpha} + O^1(T)p^- + O^1(T)p^+ + O^2(\theta, L) + \dots \\
&\{L_\alpha, L_\beta\} = O^1(T) + O^1(L)O^1(\theta, L) + \dots
\end{aligned} \tag{4.16a}$$

$$\begin{aligned}
\{x^m, \theta^\alpha\} = {} & \underline{\theta^\delta[\Gamma^m\tilde{X}(a_1p^+ + a_2p^-)} + \pi^m(\tilde{\tilde{a}}_1p^+ + \tilde{\tilde{a}}_2p^-)]_\delta{}^\alpha + \\
& + \underline{L_\delta[\tilde{\Gamma}^m(b_1p^+ + b_2p^-)} + \pi^m\tilde{X}(\tilde{\tilde{b}}_1p^+ + \tilde{\tilde{b}}_2p^-)]^{\delta\alpha} + O^1(T)O^1(\theta, L) + O^3(\theta, L) + \dots \\
\{x^m, L_\alpha\} = {} & \underline{\theta^\delta[\Gamma^m(c_1p^+ + c_2p^-)} + \pi^m X(\tilde{\tilde{c}}_1p^+ + \tilde{\tilde{c}}_2p^-)]_{\delta\alpha} + \\
& + \underline{L_\delta[\Gamma^m\tilde{X}(d_1p^+ + d_2p^-)} + \pi^m(\tilde{\tilde{d}}_1p^+ + \tilde{\tilde{d}}_2p^-)]_\alpha{}^\delta + O^1(T)O^1(\theta, L) + O^3(\theta, L) + \dots \\
\{\pi^m, \theta^\alpha\} = {} & O^1(T) + O^1(\theta, L) + \dots \\
\{\pi^m, L_\alpha\} = {} & O^1(L) + O^1(T)O^1(\theta, L) + O^1(L)O^2(\theta, L) + \dots
\end{aligned} \tag{4.16b}$$

$$\begin{aligned}
\{\pi^m, \pi^n\} = {} & O^1(T) + O^2(\theta, L) + \dots \\
\{x^m, \pi_n\} = {} & \underline{\delta^m{}_n} + O^1(T) + O^2(\theta, L) + \pi^m\pi_n G + \dots \\
\{x^m, x^n\} = {} & \underline{\frac{1}{2}\theta^\alpha\theta^\beta[-\Gamma^{[m}\tilde{X}\Gamma^{n]}g_1 + \dots]_{\beta\alpha} + \theta^\alpha L_\beta[-\tilde{\Gamma}^{[m}\Gamma^{n]}g_3 + \dots]^\beta{}_\alpha} \\
& + \underline{\frac{1}{2}L_\alpha L_\beta[-\tilde{\Gamma}^{[m}\Gamma^{n]}\tilde{X}h_1 + \dots]^{\beta\alpha}} + O^1(T) + \dots
\end{aligned} \tag{4.16c}$$

It is straightforward to check (by making use of Eqs. (3.32)) that the coefficient matrices in Eqs. (4.16) are of the most general form.

Now, we may require the Jacobi identity for different cycles. For our purposes, it

orders in θ and L carrying the free vector indices m, n on the Γ-matrices only. All another contributions are neglected. In particular, one can directly verify that only the stressed terms in Eqs. (4.16) are essential. The results of this analysis look as follows.

The $(x^m, \theta^\alpha, L_\beta)$ cycle yields in the zeroth order in θ and L

$$\frac{1}{2m} + d_1 = O^1(T), \qquad \frac{1}{2m} + \frac{c_1}{4im} + a_2 = O^1(T). \tag{4.17}$$

Analogous results for the $(x^m, \theta^\alpha, \theta^\beta)$- and (x^m, L_α, L_β)-cycles are

$$\frac{1}{8im^2} - \frac{a_1}{4im} + b_2 = O^1(T), \qquad c_2 = O^1(T). \tag{4.18}$$

Evaluating the cycle (x^m, x^n, L_α) one can get that the terms linear in θ and L vanish if the equations

$$c_1 a_2 + \frac{1}{2\sqrt{-\pi^2}} c_1 + g_1 = O^1(T), \qquad \frac{1}{2\sqrt{-\pi^2}} c_1 + c_1 d_1 = O^1(T); \tag{4.19a}$$

$$-d_1 d_2 + \frac{1}{2\sqrt{-\pi^2}}(d_1 - d_2) = O^1(T), \qquad c_1 b_2 - d_1 d_2 + \frac{1}{2\sqrt{-\pi^2}}(d_1 - d_2) + g_3 = O^1(T) \tag{4.19b}$$

were fulfilled. Comparing now the first equations in Eqs. (4.17) and (4.19b) one concludes that

$$\frac{1}{m\sqrt{-\pi^2}} = O^1(T). \tag{4.20}$$

This is a contradictory equation. Thus, for the $D = 9$ massive superparticle a possibility to continue the Jacobi identity off the constraint surface proved to be in a conflict with the manifest Poincaré covariance. The following remarks are relevant here. First, if one considers the Jacobi identity in a weak sense the contradictions do not appear (for example, Eq. (4.19b) is a coefficient at χ^- in the (x, x, L)-cycle) what reproduces the result of Sec. 3.2. Secondly, it is straightforward to check that the requirement of the Jacobi identity in the case when the coefficients at the constraints are supposed to be x^m-dependent leads to the singular fundamental phase variable brackets.

5 Conclusion

In the present paper we have constructed a consistent covariant Dirac-type bracket for the Brink–Schwarz superparticle in arbitrary dimension. This was achieved by enlarging the original phase space and introducing into the consideration a pair of strong projectors (existing for described case in the extended space only) splitting the original fermionic constraints into (infinitely) reducible first and second class parts. The proposed bracket was shown to possess all the rank and algebraic properties of a standard Dirac bracket when restricted to the second class constraints surface, what is sufficient for conventional canonical quantization of the theory. A covariant (redundant) gauge fixing and a con-

A possibility to quantize the superparticle on the basis of the "unified constraint dynamics" by Batalin and Tyutin was examined. As was shown, although the classical boundary conditions for the quantization procedure can be constructed, the covariant quantum realization of the quantities is problematic. The latter circumstance turned out to be related to the impossibility to continue off-shell the Jacobi identity for the constructed bracket along the Poincaré covariant lines.

In this paper we have realized the general procedure presented in Sec. 2 for the superparticle models. We hope, however, that the approach can be extended as well to the superstring and superbrane models which possess similar problems.

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